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Corrigendum

Additions and corrections to "Terminal coalgebras in well-founded set theory"

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Abstract

Barr, M., Additions and corrections to "Terminal coalgebras in well-founded set theory", Theoretical Computer Science 124 (1994) 189–192.

This note is to correct certain mistaken impressions of the author's that were in the original paper, "Terminal coalgebras in well-founded set theory", which appeared in *Theoretical Computer Science* 114 (1993) 299–315.

While writing the original paper, I was under certain misapprehensions as to what claims were being made in the paper of Aczel and Mendler [1]. In part this was owing to my misreading of that paper and in part to the lack of ready availability of a reference in that paper. After some discussion with Peter Aczel, I would have modified the paper as follows. First, Aczel and Mendler did not use the actual theory of non-well-founded sets to derive the proof, as I had believed. Apparently the dependence goes in the other direction; the categorical result gives an existence proof for a model of the non-well-founded sets. Second, my Theorem 1.3, as stated, is weaker than their result and not strong enough for their intended application to well-founded sets. Fortunately, the required result follows with the same proof.

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Here are the changes I would have made had I been able to before the paper actually appeared. First the title would have been "Terminal coalgebras for endofunctors on sets".

Next, the abstract should be reworded as follows:

Abstract

This paper shows that the main results of Aczel and Mendler (1989) on the existence of terminal coalgebras for an endofunctor on the category of sets do not, for the main part of the results require looking at functors on the category of (possibly proper) classes. We will see here that the main results are valid for sets up to some regular cardinal. Should that cardinal be inaccessible, then Aczel and Mendler's results are derived. In addition we discuss the canonical map from the initial algebra for an endofunctor on sets to the terminal coalgebra and show that in many cases it embeds the former as a dense subset of the latter in a certain natural topology. By way of example, we calculate the terminal coalgebra for various simple endofunctors.

The first three paragraphs of the introduction should remain as they are, but the last three should be deleted and replaced by the following.

A Set-based functor on SET is a functor $T\colon \mathbf{SET}\to\mathbf{SET}$ such that for any class X, TX is a colimit of the TA where A is a subset of X. If we interpret SET as the category of sets of cardinality up to and including some inaccessible cardinal κ and Set as the category of all those of cardinality less than κ, then a Set-based functor is the same thing as a κ-accessible functor whose value on every set of cardinality less than κ has cardinalilty at most κ. Thus Aczel and Mendler's result can thus be interpreted as saying that if κ is an inaccessible cardinal and if T is a κ-accessible functor whose value on a set of cardinality less than κ is at most κ, then there is a terminal T-coalgebra of size at most κ. (Note that "accessible" in the sense of Makkai and Paré [3] has nothing to do with accessible and inaccessible cardinals. This unfortunate clash of terms will not cause much trouble, since this paper deals only peripherally with the latter.)

Here we use an argument based on the special adjoint functor theorem, a basic tool of category theory, to show that there is a much more general construction that applies to any regular cardinal and specializes to the theorem of Aczel and Mendler when that cardinal is inaccessible.

The first paragraph of Section 1 should read as follows (this is not a change; it just adds the definition of "create").

For a set A we let $|A|$ denote the cardinality of A. We begin with a preliminary result. A functor $U\mathscr{A}\to\mathscr{B}$ is said to *create* the colimit of a diagram $D\colon\mathscr{I}\to\mathscr{A}$ is given a colimit cocone $UD \to B$ in $\mathscr{B}$, there is a unique colimit cocone $D \to A$ in $\mathscr{A}$ such that U applied to $D \to A$ gives a cocone isomorphic to $UD \to B$. This usually happens when the category $\mathscr{B}$ is a category of objects of $\mathscr{A}$ with additional structure and U is the functor that forgets that structure. There is, by the way, a similar definition for limits.

The statement of Proposition 1.3 should be modified. The only change is that a strong inequality is replaced by a weak one. However, I am indebted to Peter Aczel for pointing out to me that the original statement was not strong enough to imply the main theorem of Aczel and Mendler. In addition, Aczel pointed out that I did not use the term "weakly inaccessible" correctly. Accordingly, the paragraph preceding Proposition 1.3, the statement of the proposition and the first two paragraphs of the proof should read as follows.

We now prove the result from which the theorem of Aczel and Mendler [1] follows. Although their result is (equivalent to one) stated for inaccessible cardinals, the argument is in fact valid for any cardinal κ that is regular and for which, in addition, $\lambda < \kappa$ implies $2^\lambda \leqslant \kappa$.

Proposition. *Let $\kappa > \aleph_0$ be such a cardinal as described above and T: **Set** $\to$ **Set** be a κ-accessible functor. Suppose, in addition, that when $|A| < \kappa$, then $|TA| \leqslant \kappa$. Then* $\mathbf{Set}_T$ *has a terminal coalgebra of cardinality no larger than κ.*[1]

Proof. We claim that there is a set of generators each of cardinality less than κ. In fact, let $\alpha : A \to TA$ be a coalgebra. Since inclusions of nonempty subsets split (have right inverses), T takes inclusions to injections. It will simplify notation to suppose that T takes subsets of A to subsets of TA. From the definition of accessible, there is, for each $a \in A$, a subset $A_a \subseteq A$ such that $|A_a| < \kappa$ and $\alpha(a) \in TA_a$. Let B_0 be a subset of A of cardinality less than κ. Let $B_1 = B_0 \cup \bigcup_{a \in B_0} A_a$. Then $B_0 \subseteq B_1$ and $\alpha(B_0) \subseteq TB_1$. This is a union of fewer than κ sets, each of cardinality less than κ and since κ is regular, it follows that $|B_1| < \kappa$. In this way we can build up a countable chain of subsets.

$$B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n \subseteq \cdots$$

of subsets of B of cardinality less than κ such that $\alpha(B_n) \subseteq TB_{n+1}$. If we let $B = \bigcup B_n$, it follows that $\alpha(B) \subseteq TB$, so that B is a subcoalgebra. Since each $|B_n| < \kappa$, we have $|B| < \kappa$ as well.

Thus there is a set of generators $G_i = (A_i, \alpha_i)$ with all $|A_i| \leqslant \kappa$. The cardinality of the set of all the A_i is at most κ. For each one, there are at most $|\mathrm{Hom}(A_i, TA_i) \leqslant \kappa^\lambda$ coalgebra structures. But $\kappa^\lambda = \bigcup_{\mu < \kappa} \mu^\lambda$ since each function

[1] I would like to thank Peter Aczel for tightening up the statement of this theorem; the original form would not have implied his result.

$\lambda \to \kappa$ must, by regularity, factor through a smaller ordinal. For $\mu < \kappa$, $\mu^\lambda \leqslant (2^\mu)^\lambda = 2^{\mu \times \lambda} \leqslant \kappa$. Thus κ^λ is a union of κ sets of cardinality at most κ and hence the cardinality of this set of generators is at most κ. The coalgebra $G = \sum G_i$, whose underlying set is $\sum A_i$, has cardinality at most κ since it is the sum of at most κ many sets each of cardinality at most κ. Since the underlying functor creates colimits, it also creates epimorphisms, which are thereby surjective, and a quotient of this coalgebra also has size at most κ. We let G_0 be the colimit (cointersection) of all these quotients. Just as the intersection of subobjects of an object is still a subobject, this colimit is also an epimorphic image of G and hence $|G_0| \leqslant \kappa$.

References

[1] P. Aczel and N. Mendler, A final coalgebra theorem, in: D.H. Pitt et al., eds., *Category Theory and Computer Science*, Lecture Notes in Computer Science, Vol. 389 (Springer, Berlin, 1989) 357–365.

[2] M. Barr, Terminal coalgebras in well-founded set theory, *Theoret. Comput. Sci.* **114** (1993) 299–315.

[3] M. Makkai and R. Paré, *Accessible Categories*, Contemporary Mathematics, Vol. 104 (Amer. Math. Soc., Providence, RI, 1990).